

Confinement, Diquarks and Goldstone's theorem

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Determinations of the gluon propagator in the continuum and in lattice simulations are compared. A systematic truncation procedure for the quark Dyson-Schwinger and bound state Bethe-Salpeter equations is described. The procedure ensures the flavour-octet axial-vector Ward identity is satisfied order-by-order, thereby guaranteeing the preservation of Goldstone's theorem; and identifies a mechanism that simultaneously ensures the absence of diquarks in QCD and their presence in $\text{QCD}^{N_c=2}$, where the colour singlet diquark is the “baryon” of the theory.

The Dyson-Schwinger Equations (DSEs) provide a Poincaré covariant, nonperturbative, continuum approach to studying QCD,¹ in which the fundamental quantities are the Schwinger functions (Euclidean Green functions). The weak-coupling expansion of the DSE for a given Schwinger function generates its perturbative series. However, it is the nonperturbative nature of the DSEs that is most interesting because it entails that they provide an ideal framework for the study of confinement, dynamical chiral symmetry breaking (DCSB) and the identification of observable effects of bound state substructure in interactions involving hadrons. Recent applications of the DSEs have included the study of: meson spectroscopy;² π - π scattering;³ ω - ρ mixing and the ω - ρ mass splitting;⁴ the electromagnetic form factors of charged and neutral pions and kaons;⁵ the anomalous $\gamma^*\pi^0 \rightarrow \gamma$ ^{6,7}, $\gamma\pi^* \rightarrow \pi\pi$ ⁸ and $\gamma\pi\rho$ ⁹ transition form factors; the electroproduction of vector mesons;¹⁰ and deconfinement and chiral symmetry restoration in finite temperature QCD.¹¹

In present phenomenological applications the two most used DSEs are the *gap equation*, which yields the dressed-quark propagator, and the Bethe-Salpeter equation (BSE) for two-body bound states, which yields the mass and bound state amplitude. The primary element of the kernels of these equations is the two-point gluon Schwinger function (dressed-gluon propagator), $D_{\mu\nu}(k) = (\delta_{\mu\nu} - k_\mu k_\nu/k^2) D_T(k^2)$. The two-loop perturbative result for $D_T(k^2)$ is quantitatively reliable for $k^2 \gtrsim 1-2 \text{ GeV}^2$, however, for $k^2 < 1 \text{ GeV}^2$ nonperturbative methods are required to calculate $D_T(k^2)$.

Nonperturbative studies of the DSE for the gluon vacuum polarisation indicate a strong enhancement of $D_T(k^2)$ for $k^2 < 1 \text{ GeV}^2$, with qualitative agreement that $D_T(k^2)$ exhibits a regularised infrared (IR) singularity, which is often characterised as a regularisation of $1/k^4$.^{12,13} This is illustrated in Fig. 1. The regularisation is crucial since $D_T(k^2)$ appears in integrands sampled in

domains containing $k^2 = 0$. This behaviour is consistent with confinement because $D_{\mu\nu}(k)$ thus described: 1) doesn't have a Lehmann representation and therefore no asymptotic gluon excitation is associated with it; and 2) provides for area-law behaviour of the Wilson loop.¹⁴

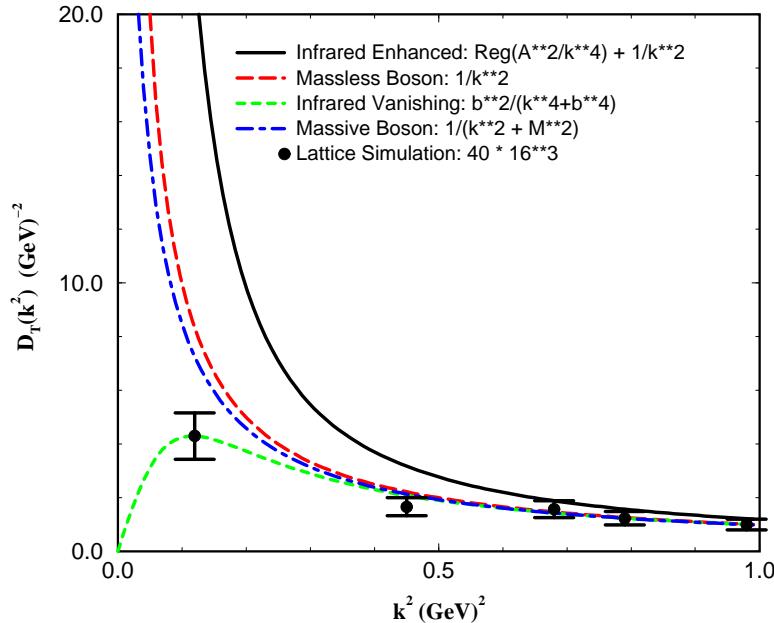


Figure 1: A comparison of possible behaviours of the gluon propagator in the IR. $A = 0.44$ GeV in the IR-enhanced gluon propagator;¹² $b = 0.34$ GeV in the IR-vanishing form;¹⁵ $M = 0.13$ GeV in the massive-boson form.¹⁶ The error-bars on the lattice points¹⁵ are only my crude estimate of the statistical error associated with 25 configurations.

The possibility that $D_T(k^2 = 0)$ might be finite or zero has been explored and found to be phenomenologically untenable because such a form does not provide for quark confinement and only leads to dynamical chiral symmetry breaking under special circumstances.¹⁷ Furthermore, recent analyses indicate that the gluon DSE does not admit such solutions.^{13,18}

One can compare these observations with numerical simulations in which the Landau gauge gluon propagator is extracted.^{15,16} In Fig. 1 results¹⁵ of a simulation with 25 gauge-configurations at $\beta = 6.0$ on a $16^3 \times 40$ -lattice are plotted. It is reported that a χ^2 -fit to all the collected data favours an IR-vanishing form of $D_T(k^2)$, although a massive boson propagator could not be eliminated from consideration. Consistent with this uncertainty, one notes in

Fig. 1 that it is only the first two data points that begin to distinguish between the forms plotted. This highlights a recognised difficulty¹⁶ in extracting the small- k^2 behaviour of $D_T(k^2)$ in lattice simulations. Finite lattice size entails that few data points can be collected at small- k^2 .

The true magnitude of this difficulty is greater, as illustrated in a study of 500 configurations at $\beta = 6.0$ on a $24^3 \times 40$ lattice,¹⁶ which finds that $D_T(k^2 = 0)$ is finite and nonzero. It is argued therein that their first 3 data points, at $k^2 < 0.34 \text{ GeV}^2$, should not be included in the fit: without them $\chi^2 \approx 1$ and it is stable; including them dramatically degrades the fit-quality. (A similar effect is observed if data at large- k^2 is included, which are affected by finite-spacing artifacts. It is argued that only the data in a window at intermediate k^2 should be included in the fitting procedure.) In the context of Fig. 1, such considerations suggest that at least the first two lattice data points can be neglected, in which case no curve is favoured over another.

In lattice simulations, the finite volume provides an intrinsic IR cutoff, which necessarily entails that the gluon propagator is finite at $k^2 = 0$.¹⁶ There is a mass-scale, M , associated with this finite value. A first estimate shows that M remains finite and nonzero in the infinite volume limit. However, the infinite volume limit alone does not indicate the behaviour of M in the continuum limit,¹⁶ which corresponds to $V \rightarrow \infty$ and $\beta \rightarrow \infty$. There is presently no data that can provide an indication of the value of M in the continuum limit. One notes that should $M \rightarrow 0$ in the continuum limit then this study¹⁶ would yield an IR-enhanced gluon propagator.

For the present the most reliable determination of the qualitative behaviour of $D_T(k^2)$ in the IR is provided by the DSE studies. One result of the IR-enhanced form of $D_T(k^2)$ is DCSB *without* fine tuning. The quark propagator is commonly written $S(p)^{-1} = i\gamma \cdot p A(p^2) + B(p^2)$ and the quark-DSE is

$$i\gamma \cdot p [A(p^2) - 1] + B(p^2) = m + \frac{4}{3} \int \frac{d^4 k}{(2\pi)^4} g^2 D_{\mu\nu}(p-k) \gamma_\mu S(k) \Gamma_\nu^g(k, p), \quad (1)$$

where $\Gamma_\nu^g(k, p)$ is the dressed-quark-gluon vertex. At any finite order in perturbation theory the dressed-quark mass function, $B(p^2)$, vanishes if the renormalised quark mass is zero; i.e., in the chiral limit. The quark condensate is nonzero if-and-only-if $B(p^2)$ is nonzero. With an IR-enhanced gluon propagator (1) necessarily admits $B(p^2)$ nonzero, even in the chiral limit; hence one has DCSB. This result does not depend on the exact form of the Ansatz used to describe the quark-gluon vertex.¹ Another result is quark confinement. $S(p)$ obtained as a solution of (1) with an IR-enhanced gluon propagator is itself enhanced in the vicinity of $k^2 = 0$ and does not have a Lehmann representation. This entails that no asymptotic quark excitation is present in the spectrum.

All studies of meson and quark-quark (diquark) bound states to date have used the rainbow-ladder truncation of the quark-DSE and meson/diquark-BSE. The rainbow quark-DSE has $\Gamma_\nu^g(k, p) = \gamma_\mu$ in (1) and the meson ladder-BSE is

$$\Gamma(p; P) = - \int \frac{d^4 q}{(2\pi)^4} g^2 D_{\mu\nu}(p - q) \gamma_\mu \frac{\lambda^a}{2} S(q_+) \Gamma(q; P) S(q_-) \gamma_\mu \frac{\lambda^a}{2}, \quad (2)$$

where $q_\pm \equiv q \pm P/2$, p is the relative quark-antiquark momentum and P is the total momentum of the meson. This pairing has been phenomenologically successful for ground state flavour-octet $[(8)_f]$ pseudoscalar, vector and axial-vector mesons, primarily because it is a Goldstone theorem preserving truncation.¹⁹ This can be seen heuristically by substituting $\Gamma_\pi^i(p; P) = \tau^i \gamma_5 [iE_\pi(p; P) + \gamma \cdot P F_\pi(p; P)]$ into (2) to obtain

$$E_\pi(p; P) = 4 \int \frac{d^4 q}{(2\pi)^4} D_T(p - q) \frac{E_\pi(q; P)}{q^2 A(q^2)^2 + B(q^2)^2} + O(P^2), \quad (3)$$

with a similar but more complicated equation for F_π . In the chiral limit, rainbow-truncation of (1)

$$B(p^2) = 4 \int \frac{d^4 q}{(2\pi)^4} D_T(p - q) \frac{B(q^2)}{q^2 A(q^2)^2 + B(q^2)^2}. \quad (4)$$

Hence, for $P^2 = 0$, the solution of (3) is $E(p; P) \propto B(p^2)$, with $F_\pi(p; P) \neq 0$ and completely determined by $A(p^2), B(p^2)$. In this truncation therefore DCSB necessarily entails, *without* fine-tuning, a massless, pseudoscalar bound state of a dressed-quark and -antiquark whose bound state amplitude is completely determined by the nonperturbative, dressed-quark propagator. (This result persists if the remaining Dirac amplitudes are retained in Γ_π^i .)

Underpinning this result is the fact that the rainbow-ladder truncation is a $(8)_f$ axial-vector Ward identity preserving truncation. The inhomogeneous ladder-BSE for the $(8)_f$ axial-vector vertex is

$$i\Gamma_\rho^5(p; P) = i\gamma_5 \gamma_\rho - \frac{4}{3} \int \frac{d^4 q}{(2\pi)^4} g^2 D_{\mu\nu}(p - q) \gamma_\mu S(q_+) i\Gamma_\rho^5(q; P) S(q_-) \gamma_\nu. \quad (5)$$

Contracting both sides with P_μ one finds that the chiral limit axial-vector Ward identity: $-iP_\mu \Gamma_\mu^5(k; P) = S^{-1}(k_+) \gamma_5 + \gamma_5 S^{-1}(k_-)$, is satisfied if-and-only-if $S(p)$ is the solution of the rainbow quark-DSE. (In considering renormalisation this heuristic outline acquires some subtleties but the result is qualitatively unchanged.) The conclusion is that Goldstone's theorem is manifest in any DSE truncation scheme that preserves the $(8)_f$ axial-vector Ward identity.

One systematic procedure for constructing such a scheme is based on a skeleton graph expansion of the dressed quark-gluon vertex in (1).²⁰ In this skeleton expansion every line and vertex is considered to be fully dressed *except* the quark-gluon vertex, which is bare. It is easiest explained via illustration. The first term, $O(g^2)$, yields the rainbow quark-DSE. Consider the integrand in (1) with $\Gamma_\nu^g(k, p) = \gamma_\mu$: the replacement $\mathcal{R} \equiv \gamma_\mu S(k) \gamma_\nu \rightarrow \gamma_\mu S(k_+) \Gamma(k; P) S(k_-) \gamma_\nu$ yields the ladder kernel for the meson BSE. This pair of equations is a $(8)_f$ axial-vector Ward identity preserving truncation.

In this expansion the $O(g^4)$ contribution to the quark-DSE is

$$2g^4 \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 l}{(2\pi)^4} \left\{ \frac{1}{9} D_{\mu\nu}(p-k) D_{\rho\sigma}(p-l) \gamma_\mu S(k) \gamma_\rho S(l+k-p) \gamma_\nu S(l) \gamma_\sigma \right. \\ \left. + i V_{\alpha\beta\gamma}(k, l, p) D_{\mu\alpha}(p-k) D_{\nu\beta}(k-l) D_{\rho\gamma}(l-p) \gamma_\mu S(k) \gamma_\nu S(l) \gamma_\gamma \right\}, \quad (6)$$

where $V_{\alpha\beta\gamma}(k, l, p)$ is the dressed 3-gluon vertex. Performing the replacement \mathcal{R} sequentially at the site of each S in (6) yields, from line one, 2 quark-gluon vertex correction terms and 1 crossed-box term, and from line two, 2 3-gluon vertex terms. These are the 5 contributions to the meson BSE kernel that are sufficient and necessary at this order to ensure that the $(8)_f$ axial-vector Ward identity is satisfied, which ensures that Goldstone's theorem is preserved at this order *without* fine tuning. This procedure can be continued order-by-order.

The ladder BSE is purely attractive in the colour-singlet $[(1)_c]$ pseudoscalar meson channel. Repulsive terms only occur at $O(g^4)$: the quark-gluon-vertex correction terms obtained from (6) via \mathcal{R} are attractive; the crossed box term is repulsive; one of the 3-gluon-vertex contributions is attractive, the other repulsive. A simple heuristic study²⁰ shows that, at $O(g^4)$, the attractive contributions almost completely cancel the repulsive terms. The terms themselves are not small but their sum is. A persistence of this cancellation order-by-order explains the success of the rainbow-ladder DSE-BSE pairing for the $(8)_f$ pseudoscalar, vector and axial-vector mesons. In the scalar sector all $O(g^4)$ are repulsive and there is no cancellation. This explains the failure of the rainbow-ladder pairing for scalar mesons: they are not simply ladder bound states of a dressed-quark and -antiquark.

Given the meson BSE it is straightforward to obtain the analogous diquark equation.²⁰ If a solution of this equation exists then the QCD spectrum contains coloured bound states, either colour-antitriplet or -sextet; colour confinement entails that no such solutions should exist. The diquark ladder-BSE is

$$\Gamma_D(p; P) = - \int \frac{d^4 q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q) \gamma_\mu \frac{\lambda^a}{2} S(q_+) \Gamma_D(q; P) \left[\gamma_\mu \frac{\lambda^a}{2} S(-q_-) \right]^T, \quad (7)$$

where T indicates matrix-transpose. (7) binds colour-antitriplet diquark bound states in QCD.² (There are no solutions of (7) in the colour-sextet sector just as there are no colour-octet solutions of (2).) This failing is due to the purely attractive nature of the kernel described above.

It is in this connection that the repulsive terms appearing at $O(g^4)$ and higher are significant. The algebra of $SU(3)_{\text{colour}}$ entails that the two $O(g^4)$ 3-gluon vertex diagrams still contribute with opposite signs in the diquark equation, however, relative to the vertex correction contributions, the repulsive $O(g^4)$ crossed-box term in the diquark equation is five times stronger than in the meson equation. The repulsive effect of this term is further amplified by the IR enhancement of $S(p)$. These effects together act to ensure there is no solution of the $O(g^4)$ diquark BSE. The persistence of this effect at higher order would explain the absence of diquark bound states in QCD.

A qualitative check of this mechanism and systematic truncation procedure is found in $\text{QCD}^{N_c=2}$, where the $(1)_c$ “baryon” is a diquark. A truncation procedure that, order-by-order, ensures the absence of coloured diquark bound states in QCD should simultaneously ensure the existence of $(1)_c$ diquarks (baryons) in $\text{QCD}^{N_c=2}$.

The generators of $SU(2)$ are $\{\tau^i/2\}_{i=1\dots 3}$, where $\vec{\tau}$ are the Pauli matrices. The colour structure of the $(1)_c$ meson is described by the 2×2 -identity matrix and that of the diquark by $i\tau^2$. At a given order the meson BSE equation will involve strings of the form

$$\gamma_{\mu_n} \tau^{i_n} S(k_n) \dots \gamma_{\mu_{j+1}} \tau^{i_{j+1}} S(k_{j+1}) \Gamma_M(q; P) S(k_j) \gamma_{\mu_j} \tau^{i_j} \dots S(k_1) \gamma_{\mu_1} \tau^{i_1} \quad (8)$$

while the associated term in the diquark equation will be

$$\begin{aligned} & \gamma_{\mu_n} \tau^{i_n} S(k_n) \dots \\ & \gamma_{\mu_{j+1}} \tau^{i_{j+1}} S(k_{j+1}) \Gamma_D(q; P) i\tau^2 [\gamma_{\mu_j} \tau^{i_j} S(-k_j)]^T \dots [\gamma_{\mu_1} \tau^{i_1} S(-k_1)]^T. \end{aligned} \quad (9)$$

Defining $\Gamma_D^C = \Gamma_D C^\dagger$, where $C = \gamma_2 \gamma_4$, and using: $\tau^i \tau^2 [\tau^j]^T = -\tau^i \tau^j \tau^2$; and $[\gamma_\mu]^T = -C^\dagger \gamma_\mu C$, (9) becomes

$$\gamma_{\mu_n} \tau^{i_n} S(k_n) \dots \gamma_{\mu_{j+1}} \tau^{i_{j+1}} S(k_{j+1}) \Gamma_D^C(q; P) S(k_j) \gamma_{\mu_j} \tau^{i_j} \dots S(k_1) \gamma_{\mu_1} \tau^{i_1} \tau^2. \quad (10)$$

This demonstrates that, in $\text{QCD}^{N_c=2}$, $\Gamma_M(q; P)$ and $\Gamma_D^C(q; P)$ satisfy the same equation, order-by-order. The truncation scheme and diquark confinement mechanism therefore satisfy the constraint indicated above. In fact, one sees that in $\text{QCD}^{N_c=2}$ the spectrum of mesons and baryons (diquarks) is identical (neglecting electroweak effects). In particular, the existence of a pseudoscalar Goldstone boson entails the existence of a massless scalar baryon (diquark).

These results are simply a manifestation of the equivalence of the fundamental and conjugate representations of $SU(2)$.

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